

Fluctuation Susceptibility Relations for Classical Spin Systems

Joël De Coninck¹ and François Dunlop²

Received September 7, 1984; revised February 12, 1985

We prove identities between integrated Ursell functions and derivatives of the pressure in the thermodynamic limit, for multicomponent classical spin systems which obey the Lee–Yang theorem and some form of Gaussian domination, when the susceptibility is finite ($T > T_c$). Following Refs. 3 and 4, we view the moment generating function of the magnetization as the inverse of an infinitely divisible characteristic function. Fluctuation susceptibility relations of all orders then follow by bounding the corresponding cumulants, taken in zero external field. High-order cumulants are bounded in terms of the susceptibility using Gaussian and Simon's inequalities for short-range interactions.

KEY WORDS: Fluctuation susceptibility relations; Lee–Yang; infinitely divisible distribution; mass gap.

1. INTRODUCTION

As already pointed out in Ref. 1, a remarkable consequence of the Lee–Yang theorem is the proof of “fluctuation susceptibility relations” at nonzero external field, i.e., that the finite volume cumulants of the magnetization variable converge to the corresponding derivatives of the pressure, when the external field is nonzero. The proof extends to zero external field only when there is a “Lee–Yang gap,” but the existence of this gap has been established only for high enough temperatures⁽²⁾ and for some particular models.

Although the critical temperature could be defined as the temperature where the Lee–Yang gap vanishes, it is more usual to define it as the tem-

¹ Université de l'État à Mons, Faculté des Sciences, 7000 Mons—Belgium.

² École Polytechnique, Centre de Physique Théorique (Laboratoire propre du C.N.R.S. No. 14), 91128 Palaiseau Cedex—France.

perature where the susceptibility diverges. One may then wonder if the fluctuation susceptibility relations hold at zero external field for temperatures such that the susceptibility is finite.

This question has been addressed previously by various authors.⁽⁵⁻⁸⁾ A crucial ingredient, for dominated convergence arguments, is to obtain bounds uniformly in a neighborhood of $h=0$. Lebowitz⁽⁵⁾ gave an answer to this problem by first bounding the n -point Ursell functions, for arbitrary h , in terms of the two-point function at the same value of h (by FKG inequalities), and then bounding the two-point function at $h \neq 0$ by the same function at $h=0$ (by the GHS inequality).

Our method follows a different path: we use a consequence of the Lee-Yang theorem, for all integrated Ursell functions altogether, to reduce the problem from $h \neq 0$ to $h=0$, and then use Gaussian inequalities to bound n -point Ursell functions at $h=0$ by the two-point function at $h=0$. The first step is essentially the fact, established in Refs. 3 and 4, that integrated Ursell functions are (minus) the derivatives of the logarithm of a characteristic function (Fourier transform of a positive measure). This involves dividing by the volume, and is possible for infinitely divisible distributions.

This method has the advantage to apply also to two- or three-component spins for which FKG inequalities do not exist. The lengths of the spins may be unity (Ising, classical Heisenberg model) or continuous random variables (e.g., $|\phi|^4$ models). The geometry of the lattice is arbitrary.

2. MODELS AND NOTATION

Consider the family of finite-volume Gibbs states given by

$$d\mu_A^{\beta,h}((\sigma_j)_{j \in A}) = Z_A^{-1}(\beta, h) \exp \left\{ \beta \sum_{i,j \in A} J_{ij} \sigma_i \cdot \sigma_j + h \sum_{k \in A} \sigma_k^1 \right\} \prod_{j \in A} d\nu(\sigma_j) \quad (1)$$

where

$$\begin{aligned} A &\subset \mathbb{Z}^d, & \beta &> 0 \\ J_{ij} &= J(i-j) \geq 0, & \sum_{j \in \mathbb{Z}^d} J_{ij} &< \infty \end{aligned} \quad (2)$$

and where the free spin distribution $\nu(\sigma)$ is a positive measure on \mathbb{R}^N , $N=1, 2$, or 3 , which is rotation invariant and satisfies

$$\int \exp(b\sigma^2) d\nu(\sigma) < \infty \quad \forall b \in \mathbb{R} \quad (3)$$

$$\text{Re } h \neq 0 \Rightarrow \int \exp(h\sigma^1) d\nu(\sigma) \neq 0 \quad (4)$$

If $N = 3$, we require explicitly

$$dv(\sigma) = \delta(\sigma^2 - 1) d^3\sigma \tag{5}$$

or

$$dv(\sigma) = \exp(-\lambda |\sigma|^4 + \mu |\sigma|^2) d^3\sigma, \quad \lambda > 0, \mu \in \mathbb{R} \tag{6}$$

These conditions ensure^(19–21) the validity of the Lee–Yang theorem. (2) and (3) imply that the partition function is an entire function of h of order at most two, with some uniformity in A :

$$|Z_A(\beta, h)| < ce^{c|A||h|^2} \tag{7}$$

Let us now introduce the finite-volume pressure $P_A(\beta, h)$, the finite-volume difference of pressures $D_A(\beta, h)$, and the finite-volume cumulants $U_n^A(\beta, h)$, $n = 1, 2, \dots$:

$$\begin{aligned} P_A(\beta, h) &= |A|^{-1} \log Z_A(\beta, h) \\ D_A(\beta, h) &= P_A(\beta, h) - P_A(\beta, 0) \\ U_n^A(\beta, h) &= \frac{\partial^n}{\partial h^n} P_A(\beta, h) = \frac{\partial^n}{\partial h^n} D_A(\beta, h) \end{aligned}$$

As $A \nearrow \mathbb{Z}^d$, when the corresponding limit exists, we denote

$$\begin{aligned} P(\beta, h) &= \lim_{A \nearrow \mathbb{Z}^d} P_A(\beta, h) \\ D(\beta, h) &= \lim_{A \nearrow \mathbb{Z}^d} D_A(\beta, h) \\ U_n(\beta, h) &= \frac{\partial^n}{\partial h^n} D(\beta, h) \end{aligned}$$

3. RESULTS

For simplicity, we first give a theorem based only on the Lee–Yang Theorem and its consequences. Theorem 2 will use Gaussian inequalities and Simon’s inequality to remove the hypothesis on $U_{2n}^A(\beta, 0)$ for $n > 1$. The hypothesis on $U_2(\beta, 0)$ in the (necessary) condition $T > T_c$.

Theorem 1. Consider the model defined by Eqs. 1–6 and suppose that

$$\text{Sup}_A |U_{2p}^A(\beta, 0)| < \infty, \quad p = 1, 2, \dots, n \tag{8}$$

Then $D(\beta, h)$ is at least $2n$ times differentiable with respect to h , in particular at $h = 0$. The finite-volume cumulants of order up to $2n$ converge as $A \nearrow \mathbb{Z}^d$ to the corresponding derivatives of $D(\beta, h)$.

Proof. The proof of the convergence of $U_n^A(\beta, h)$ and of the fluctuation susceptibility relations (or sum rules⁽⁶⁾) is based on the convergence of $D_A(\beta, h)$, viewed as minus the logarithm of the characteristic function of an infinitely divisible distribution.^(3,4) We sketch the method for completeness.

As derived in Ref. 12, the Hadamard representation of an even entire function, of order at most 2, with pure imaginary zeros leads to

$$Z_A(\beta, h)/Z_A(\beta, 0) = \exp(bh^2) \prod_1^{\infty} (1 + h^2/h_j^2) \quad (9)$$

Each factor on the right-hand side of this expression is precisely the inverse of an infinitely divisible characteristic function. This implies the same property for the product, as well as for the $1/|A|$ root. That is to say, that there exists an infinitely divisible distribution function $F_A(x)$ such that

$$\left\{ \frac{Z_A(\beta, h)}{Z_A(\beta, 0)} \right\}^{1/|A|} = \frac{1}{\int \exp(ix) dF_A(x)} \quad (10)$$

Since

$$\{Z_A(\beta, h)/Z_A(\beta, 0)\}^{1/|A|} = \exp[D_A(\beta, h)]$$

one indeed obtains that the finite-volume difference of pressures $D_A(\beta, h)$ may be viewed as minus the logarithm of an infinitely divisible characteristic function. The infinite volume limit of $p_A(\beta, h)$, and consequently of $D_A(\beta, h)$, with free boundary conditions as in (1) is easy using (7) and, e.g., the first Griffiths inequality. The limit is continuous in h because it is convex in h .

The well-known Lévy theorem⁽¹⁴⁾ ensures that a function, continuous at the origin, which is the limit of a sequence of characteristic functions is itself a characteristic function. The corresponding sequence of probability distributions $(F_A(x))_A$ converges then weakly to the probability distribution, say, $F(x)$, of the limiting characteristic function.

In order to prove that the successive derivatives of $D_A(\beta, h)$ taken at zero external field converge, as $A \uparrow \mathbb{Z}^d$, to those of $D(\beta, h)$ one only has to prove that the sequence of cumulants of $(F_A(x))_A$ converge, as $A \uparrow \mathbb{Z}^d$, to the cumulants of $F(x)$. But we have proved that $(F_A(x))_A$ converges weakly to $F(x)$. The corresponding convergence of the cumulants of $F_A(x)$ or the

equivalent fluctuation-susceptibility relations (or sum rules) now follow easily from the dominated convergence theorem, provided the moments of $F_A(x)$ of degree up to $2n$ are uniformly bounded. The desired bounds follow from (8) and Hölder's inequality.

This concludes the proof of theorem 1. ■

Remark. The bounds for $U_{2p}^A(\beta, 0)$, $1 < p < n$, follow^(3,4) from the bounds for $U_2^A(\beta, 0)$ and $U_{2n}^A(\beta, 0)$

Theorem 2. Consider the models defined by Eqs. 1–6 where for $N = 1, 2$ we suppose in addition that

$$dv(\sigma) = e^{-v(\sigma^2)} d\sigma, \quad v \text{ convex} \tag{11}$$

or a limit of such measures or a measure constructed out of (11) by the analog spin method of Griffiths.

Assume, for some $a > 0$,

$$J_{ij} < e^{-a|i-j|}, \quad i \neq j \tag{12}$$

and

$$\text{Sup}_A U_2^A(\beta, 0) < \infty \tag{13}$$

Then $D(\beta, h)$ is C^∞ in h and the finite-volume cumulants of all orders converge as $A \nearrow \mathbb{Z}^d$ to the corresponding derivatives of $D(\beta, h)$.

Proof. Using the method developed in Theorem 1, we only have to construct bounds independent of the volume for the cumulants. The bounds are obtained by combining the next two lemmas.

We first need some notations: for A a set of indices in \mathbb{Z}^d , we denote

$$\sigma_A^1 = \prod_{i \in A} \sigma_i^1$$

and $\langle \sigma_A^1 \rangle^A$ will be the expectation of σ_A^1 in the probability measure (1). The Ursell functions for the first component of the spin are defined inductively by

$$\langle \sigma_{x_1}^1 \cdots \sigma_{x_n}^1 \rangle_A = \sum_{\pi \in \mathcal{P}(\{1, \dots, n\})} \prod_{\gamma \in \pi} u_{|\gamma|}^A(\{x_i\}_{i \in \gamma})$$

where $\mathcal{P}(\{1, \dots, n\})$ is the set of partitions of $\{1, \dots, n\}$. The Ursell functions are related to the cumulants by

$$U_n^A(\beta, h) = \frac{1}{|A|} \sum_{\substack{x_1 \in A \\ \vdots \\ x_n \in A}} u_n^A(x_1 \cdots x_n)$$

Lemma 1.^(9–11) Consider a classical spin system defined by Eqs. (1) and (2) with $h=0$ and (5) and (6) or (11). Assume (12) and (13). Then there is a mass gap, i.e., $\exists m > 0$ and c such that

$$u_2(i, j) < c \exp(-m |i - j|)$$

Proof. Our formulation differs slightly from the results of Refs. 9–11. A direct proof follows the proof of Corollary 4.2 in Ref. 9. One should find a volume $\Omega \subset \mathbb{Z}^d$, $\Omega \ni i$, and a constant $a > 0$, such that

$$\beta \left(\sum_k J_{ik} e^{a|i-k|} \right) \sum_{l \in \Omega} e^{-a \text{dist}(l, \Omega^c)} u_2(i, l) < 1 \tag{14}$$

We first choose a volume $\Omega_1 \ni i$ such that

$$\sum_{l \in \Omega_1^c} u_2(i, l) < \frac{1}{2} \beta^{-1} \left(\sum_k J_{ik} e^{a|i-k|} \right)^{-1}$$

This is possible because the susceptibility is finite. We then choose $\Omega \supset \Omega_1$ such that

$$\sum_{l \in \Omega_1} e^{-a \text{dist}(l, \Omega^c)} u_2(i, l) < \frac{1}{2} \beta^{-1} \left(\sum_k J_{ik} e^{a|i-k|} \right)^{-1}$$

Then $\Omega = \Omega_1 \cup (\Omega \setminus \Omega_1)$ satisfies (14). The rest of the proof is as in Ref. 9.

Lemma 2.⁽⁵⁾ Consider a classical spin system defined by Eqs. (1) and (2) with $h=0$ and (5) and (6) or (12). Suppose $\exists c_p$ independent of A , $p=0, 1, 2, \dots$ such that

$$|A|^{-1} \sum_{x, y \in A} |x - y|^p u_2^A(x, y) < c_p, \quad p = 0, 1, 2, \dots \tag{15}$$

Then $\exists d_n$ independent of A , $n = 1, 2, \dots$ such that

$$|U_{2n}^A(\beta, 0)| < d_n \tag{16}$$

Proof. Our hypotheses (\Rightarrow Gaussian inequalities^(15,16,22)) differ from Lebowitz's (\Rightarrow FKG inequalities), but the idea is essentially the same. We first prove a bound on the Ursell function $u_{2n}^A(x_1, \dots, x_{2n})$. Let

$$d(x_1, \dots, x_{2n}) = \text{Max}_{\{1, \dots, 2n\} = A \cup B} \text{Min}_{\substack{i \in A \\ j \in B}} |x_i - x_j| \tag{17}$$

Then $\exists b_n$ independent of A such that

$$|u_{2n}^A(x_1, \dots, x_{2n})| < b_n \quad \text{Max}_{\substack{i,j \\ |x_i - x_j| \geq d(x_1, \dots, x_{2n})}} u_2^A(x_i, x_j) \quad (18)$$

This may be proven using Gaussian inequalities.

Let us now label x_1, x_2 the two points giving the Max in (18). In order to sum over $x_3 \cdots x_{2n}$, we can construct a tree graph connecting $x_1 \cdots x_{2n}$, all segments of which satisfy

$$|x_k - x_l| \leq |x_1 - x_2| \quad (19)$$

This choice of a tree graph depends on the configuration, but we include the number of choices and their overlap in the constant d_n . It follows that

$$|U_{2n}^A(\beta, 0)| \leq |A|^{-1} \sum'_{|x_k - x_l| \leq |x_1 - x_2|} b'_n u_2^A(x_1, x_2)$$

where \sum' is subject to a given tree graph. Summation over $x_3 \cdots x_{2n}$ starting from the end points of the graph, gives

$$|U_{2n}^A(\beta, 0)| \leq |A|^{-1} \sum_{x_1, x_2 \in A} b''_n |x_1 - x_2|^{(2n-2)d} u_2(x_1, x_2)$$

Hypothesis (15) concludes the proof of the lemma.

Remark. What is really wanted, in place of Lemma 2, is skeleton inequalities for all Ursell functions, which would give

$$|U_{2n}^A(0, \beta)| \leq c^n (2n)! [U_2^A(0, \beta)]^{qn} \quad (20)$$

with c and q independent of A . This would allow to suppress the short-range interactions hypothesis in Theorem 2.

Appropriate skeleton inequalities have been proven for $n=2$ by Aizenman⁽¹⁷⁾ and by Brydges, Fröhlich, Sokal.⁽¹⁸⁾ Fluctuation susceptibility relations are thus proven up to the fourth cumulant when the susceptibility is finite regardless of the range of the interactions.

ACKNOWLEDGMENTS

JDC and FD are grateful, respectively, to Centre de Physique Théorique, Ecole Polytechnique, and Faculté des Sciences, Université de l'Etat à Mons, for hospitality.

REFERENCES

1. C. N. Yang and T. D. Lee, *Phys. Rev.* **87**:404 (1952); T. D. Lee and C. N. Yang, *Phys. Rev.* **87**:410 (1952).
2. G. Gallavotti, S. Miracle-Sole, and D. W. Robinson, *Phys. Lett.* **25A**:493 (1967).
3. J. De Coninck and Ph. de Gottal, *J. Stat. Phys.* **36**:181 (1984).
4. J. De Coninck, *Commun. Math. Phys.* **96**:373 (1984).
5. J. L. Lebowitz, *Commun. Math. Phys.* **28**:313 (1972).
6. A. D. Sokal, *J. Stat. Phys.* **25**:25 (1981).
7. D. B. Abraham, *Phys. Lett.* **43A**:163 (1973).
8. M. E. Fisher, *Phys. Rev.* **180**:594 (1969).
9. B. Simon, *Commun. Math. Phys.* **77**:111 (1980).
10. M. Aizenman and B. Simon, *Commun. Math. Phys.* **77**:137 (1980).
11. J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View* (Springer, New York, 1981) p. 317; *Commun. Math. Phys.* **52**:203 (1977).
12. C. M. Newman, *Commun. Math. Phys.* **41**:1 (1975).
13. R. B. Griffiths, Rigorous results and theorems, in *Phase Transitions and Critical Phenomena*, Vol. 1, C. Domb and M. S. Green, eds. (Academic Press, New York, 1972).
14. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II (John Wiley & Sons, New York, 1971), p. 251.
15. C. M. Newman, *Z. Wahrscheinlichkeitstheor.* **33**:75 (1975).
16. D. Brydges, J. Fröhlich, and T. Spencer, *Commun. Math. Phys.* **83**:123 (1982).
17. M. Aizenmann, *Commun. Math. Phys.* **86**:1 (1982).
18. D. Brydges, J. Fröhlich, and A. D. Sokal, *Commun. Math. Phys.* **91**:117 (1983).
19. C. M. Newman, *Commun. Pure Appl. Math.* **27**:143 (1974).
20. E. H. Lieb and A. D. Sokal, *Commun. Math. Phys.* **80**:153 (1981).
21. F. Dunlop and C. M. Newman, *Commun. Math. Phys.* **44**:223 (1975).
22. J. Bricmont, *J. Stat. Phys.* **17**:289 (1977).